

The Forcing Edge Chromatic Number of Some Standard Graphs

R. Suganya and *V. Sujin Flower

Department of Mathematics, Holy Cross College (Autonomous), Nagercoil - 629004.

Affiliated to Manonmaniam Sundaranar University, Tirunelveli - 627012.

*Corresponding Author - Email: sujinflower@gmail.com

ABSTRACT

Let S be a χ' -set of G . A subset $T \subseteq S$ is said to be a forcing subset for S if S is the unique χ' -set containing T . The forcing edge chromatic number $f_{\chi'}(S)$ of S in G is the minimum cardinality of a forcing subset for S . The forcing edge chromatic number $f_{\chi'}(G)$ of G is the smallest forcing number of all χ' -sets of G . In this article, some general properties satisfied by this concept are studied and the forcing edge chromatic number of some standard graphs are determined. Also, connected graphs of order $n \geq 2$ edge chromatic number 0 or 1 or $\chi'(G)$ are characterized. It is shown that for a positive integer $a \geq 2$, there exists a connected graph G such that $f_{\chi'}(G) = \chi'(G) = a$.

Keywords: Forcing edge chromatic number, Edge chromatic number, Chromatic number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices u and v are said to be adjacent if uv is an edge of G . Two edges of G are said to be adjacent if they have a common vertex.

A k -coloring of G is a function $c : V(G) \rightarrow \{1, 2, \dots, k\}$, where $c(u) \neq c(v)$ for any two adjacent vertices u and v in G . A p -vertex coloring of G is an assignment of p colors, $1, 2, \dots, p$ to the vertices of G , the coloring is proper if no two distinct adjacent vertices have the same color. The minimum colours needed to colour the vertices of G is called *chromatic number* of G , denoted by $\chi(G)$. If $\chi(G) = p$, G is said to be p -chromatic, where $p \leq k$. A set $C \subseteq V(G)$ is called *chromatic set* if C contains all vertices of distinct colors in G . The chromatic number of G is the minimum cardinality among all the chromatic sets of G . That is $\chi(G) = \min\{|C_i| / C_i \text{ is a chromatic set of } G\}$. The concept of the chromatic number was studied in [2,3,4]. A k -edge coloring of G is a function $c' : E(G) \rightarrow \{1, 2, \dots, k\}$, where $c'(e) \neq c'(f)$ for any two adjacent edges e and f in G . A p -edge coloring of G is an assignment of p colors, $1, 2, \dots, p$ to the edges of G , the coloring is proper if no two distinct adjacent edges have the same color. The minimum colours needed to colour the edges of G is called *edge chromatic number* of G , denoted by $\chi'(G)$. If $\chi'(G) = p$, G is said to be p -edge chromatic, where $p \leq k$. A set $C' \subseteq E(G)$ is called *edge chromatic set* if C' contains all

edges of distinct colors in G . The *edge chromatic number* of G is the minimum cardinality among all the edge chromatic sets of G . That is $\chi'(G) = \min\{|C'_i| \mid C'_i \text{ is a edge chromatic set of } G\}$. An edge chromatic set of cardinality $\chi'(G)$ is called a χ' -set of G . The edge-chromatic number $\chi'(G)$ of G is defined to be the least number of colours needed to colour the edges of G in such a way that no two adjacent edges have the same colour. The concept of the edge chromatic number was studied in [5,6,7]. The chromatic number has application in Time Table Scheduling, Map coloring, channel assignment problem in radio technology, town planning, GSM mobile phone networks etc.[8,9].

2. The forcing edge chromatic number of some standard graphs

Definition 2.1. Let S be a χ' -set of G . A subset $T \subseteq S$ is said to be a forcing subset for S if S is the unique χ' -set containing T . The *forcing edge chromatic number* $f_{\chi'}(S)$ of S in G is the minimum cardinality of a forcing subset for S . The forcing edge chromatic number $f_{\chi'}(G)$ of G is the smallest forcing number of all χ' -sets of G .

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{e_1, e_2, e_3, e_4\}, S_2 = \{e_6, e_2, e_3, e_4\}, S_3 = \{e_1, e_5, e_3, e_4\}, S_4 = \{e_6, e_5, e_3, e_4\}$ are the only two χ' -sets of G such that $\chi'(G) = 3, f_{\chi'}(S_1) = f_{\chi'}(S_2) = f_{\chi'}(S_3) = f_{\chi'}(S_4) = 2$ so that $f_{\chi'}(G) = 2$.

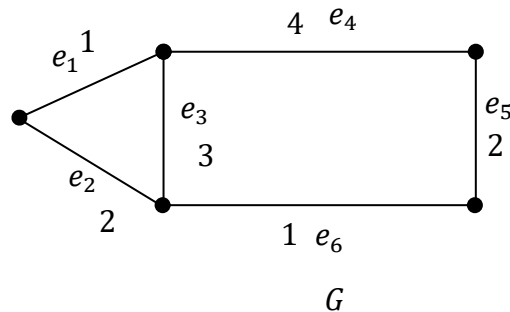


Figure 2.1

The following result follows immediately from the definitions of the edge chromatic number and the forcing edge chromatic number of a connected graph G .

Observation 2.3. For every connected graph $G, 0 \leq f_{\chi'}(G) \leq \chi'(G)$.

Remark 2.4. The bounds in the Observation 2.3 are sharp. For the complete graph $G = K_3, S = E(G)$ is the unique χ' -set of G so that $f_{\chi'}(G) = 0$. For the graph G given in Figure 2.2, $S_1 = \{e_1, e_2, e_3\}, S_2 = \{e_1, e_4, e_5\}, S_3 = \{e_1, e_2, e_6\}, S_4 = \{e_1, e_4, e_6\}, S_5 = \{e_3, e_2, e_5\}, S_6 = \{e_3, e_4, e_5\}, S_7 = \{e_3, e_2, e_6\}, S_8 = \{e_3, e_4, e_6\}$ such that $f_{\chi'}(S_i) = 3$ for $i = 1$

to 8 and $\chi'(G) = 3$ so that $f_{\chi'}(G) = \chi'(G) = 3$. Also the bounds are strict. For the graph in Figure 2.1, $\chi'(G) = 4, f_{\chi'}(G) = 2$. Thus $0 < f_{\chi'}(G) < \chi'(G)$.

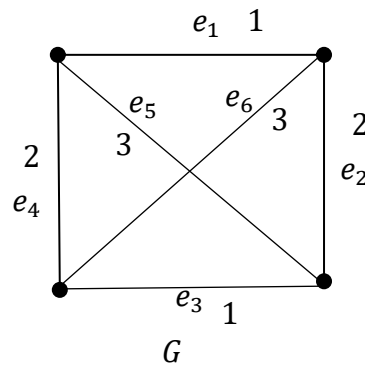


Figure 2.2

Definition: 2.5. An edge e of a graph G is said to be an *edge chromatic edge* of G if e belongs to every χ' -set of G .

Example 2.6. For the graph G given in Figure 2.3, $S_1 = \{e_1, e_2, e_3\}, S_2 = \{e_1, e_2, e_4\}, S_3 = \{e_5, e_2, e_3\}, S_4 = \{e_5, e_2, e_4\}$ are the only χ' -sets of G such that e_2 is a chromatic edge of G .

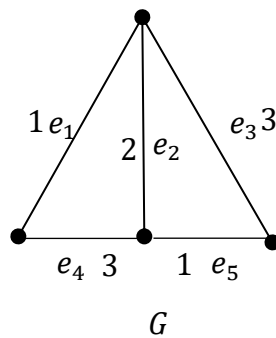


Figure 2.3

Theorem 2.7. Let G be a connected graph of order $n \geq 2$ with $\Delta(G) = n - 1$. Let x be a universal vertex of G and e be an edge incident with x . Then e is a chromatic edge of G .

Proof. On the contrary, suppose that e is not a chromatic edge of G . Then there exists a χ' -set S of G such that $e = uv$. Let $c(e) = c_1$. Since $e \notin S$, there exists $f = yz \in E(G)$ such that $c(f) = c_1$ and $y \neq x, v$ and $z \neq x, v$. Hence it follows that x is not a universal vertex of G , which is a contradiction. Therefore e is a chromatic edge of G .

Theorem 2.8. Let G be a connected graph. Then

- (a) $f_{\chi'}(G) = 0$ if and only if G has a unique χ' -set.
- (b) $f_{\chi'}(G) = 1$ if and only if G has at least two χ' -sets, one of which is a

unique χ' -set containing one of its elements, and

(c) $f_{\chi'}(G) = \chi'(G)$ if and only if no χ' -set of G is the unique χ' -set

containing any of its proper subsets.

Proof. (a) Let $f_{\chi'}(G) = 0$. Then, by definition, $f_{\chi'}(S) = 0$ for some χ' -set S of G so that the empty set \emptyset is the minimum forcing subset for S . Since the empty set \emptyset is a subset of every set, it follows that S is the unique χ' -set of G . The converse is clear.

(b) Let $f_{\chi'}(G) = 1$. Then by Theorem 2.8(a), G has at least two χ' -sets. Also, since $f_{\chi'}(G) = 1$, there is a singleton subset T of a χ' -set S of G such that T is not a subset of any other χ' -set of G . Thus S is the unique χ' -set containing one of its elements. The converse is clear.

(c) Let $f_{\chi'}(G) = \chi'(G)$. Then $f_{\chi'}(S) = \chi'(G)$ for every χ' -set S in G . Also, by Observation 2.3, $\chi'(G) \geq 2$ and hence $f_{\chi'}(G) \geq 2$. Then by Theorem 2.8(a), G has at least two χ' -sets and so the empty set \emptyset is not a forcing subset for any χ' -set of G . Since $f_{\chi'}(S) = \chi'(G)$, no proper subset of S is a forcing subset of S . Thus no χ' -set of G is the unique χ' -set containing any of its proper subsets. Conversely, the data implies that G contains more than one χ' -set and no subset of any χ' -set S other than S is a forcing subset for S . Hence it follows that $f_{\chi'}(G) = \chi'(G)$.

Theorem 2.9. Let G be a connected graph and W be the set of all chromatic edges of G . Then $f_{\chi'}(G) \leq \chi'(G) - |W|$.

Proof. Let S be any χ' -set of G . Then $\chi'(G) = |S|$, $W \subseteq S$ and S is the unique χ' -set containing $S - W$. Thus $f_{\chi'}(G) \leq |S - W| = |S| - |W| = \chi'(G) - |W|$.

In the following we determine the forcing edge chromatic number of some standard graphs.

Theorem 2.10. For the complete graph $G = K_n (n \geq 2)$,

$$f_{\chi'}(G) = \begin{cases} 0 & \text{if } n = 2, 3 \\ 3 & \text{if } n = 4 \\ n - 1 & \text{if } n \geq 5 \end{cases}$$

Proof. For $n = 2$ and $n = 3$, $S = E(G)$ is the unique χ' -set of G , the result follows from Theorem 2.8 (a). For $n = 4$, let $e_{11} = v_1v_2$, $e_{12} = v_1v_3$, $e_{13} = v_1v_4$, $e_{21} = v_2v_3$, $e_{22} = v_2v_4$, $e_{31} = v_3v_4$. Assign $c'(e_{11}) = c'(e_{31}) = 1$, $c'(e_{12}) = c'(e_{22}) = 2$, $c'(e_{13}) = c'(e_{21}) = 3$. Then $S_1 = \{e_{11}, e_{12}, e_{13}\}$, $S_2 = \{e_{11}, e_{12}, e_{21}\}$, $S_3 = \{e_{11}, e_{22}, e_{13}\}$, $S_4 = \{e_{11}, e_{22}, e_{21}\}$, $S_5 = \{e_{31}, e_{12}, e_{13}\}$, $S_6 = \{e_{31}, e_{12}, e_{21}\}$, $S_7 = \{e_{31}, e_{22}, e_{13}\}$, $S_8 =$

$\{e_{31}, e_{22}, e_{21}\}$ are the χ' -set of G such that $f_{\chi'}(S_i) = 3$ for $i = 1$ to 8 so that $f_{\chi'}(G) = 3$. For $n \geq 5$, let $e_{1j} = v_1v_j(2 \leq j \leq n), e_{2j} = v_2v_j(3 \leq j \leq n), e_{3j} = v_3v_j(4 \leq j \leq n), \dots, e_{(n-1)j} = v_{n-1}v_n$. Assign $c'(e_{1j}) = c'_j, c'(e_{2j}) = c'_j - 1 (1 \leq j \leq n - 1), c'(e_{3j}) = c'_j - 2 (1 \leq j \leq n - 1), \dots, c'(e_{(n-1)j}) = c'_j - (n - 2) (1 \leq j \leq n - 1), c'(e_{(n-2)j}) = n$ so that $\chi'(G) = n$. Since $e_{(n-2)j}$ is a chromatic edge of G , by Theorem 2.9, $f_{\chi'}(G) \leq n - 1$. Let S be a chromatic edge set of G . We prove that $f_{\chi'}(G) = n - 1$. On the contrary, suppose that $f_{\chi'}(G) \leq n - 2$. Then there exists a forcing subset T of S such that $|T| \leq n - 2$. Let $e \in S$ such that $e \notin T$. Then e is not a chromatic edge of G . Without loss of generality, let us assume that $c'(e) = c'_1$. Since $n \geq 5$, there exists $f \in E(G)$ such that $c'(f) = c'_1$. Let $S' = [S - \{e\}] \cup \{f\}$. Then S' is a χ' -set of G . Hence T is a proper subset of a χ' -set S' of G , which is a contradiction. Therefore $f_{\chi'}(G) = n - 1$.

Theorem 2.11. For the star graph $G = K_{1,n-1} (n \geq 3), f_{\chi'}(G) = 1$.

Proof. Since $S = E(G)$ is the unique χ' -set of G , the result follows from Theorem 2.8(a)

Theorem 2.12. For the double star graph $G = K_{2,r,s}, f_{\chi'}(G) = 2$.

Proof. Let $V = \{x, v_1, v_2, \dots, v_r\} \cup \{y, u_1, u_2, \dots, u_s\}$ be the vertex set of G . Let $f_i = xv_i, e = xy, g_i = yu_j$ be the edge set of G for all $(1 \leq i \leq r)$ and $(1 \leq j \leq s)$ where $r + s = n - 2$. Then $S_1 = \{e, f_i\} (1 \leq i \leq r)$ and $S_2 = \{e, g_j\} (1 \leq j \leq s)$ are the only χ' -sets of G such that $f_{\chi'}(S_1) = f_{\chi'}(S_2) = 2$ so that $f_{\chi'}(G) = 2$.

Theorem 2.13. For the complete bipartite graph $G = K_{r,s} (1 \leq r \leq s),$

$$f_{\chi'}(G) = \begin{cases} 0 & \text{if } r = 1, s = 1 \\ 2 & \text{if } r = 2, s = 2 \\ s & \text{if } 2 \leq r \leq s \end{cases}$$

Proof. For $r = 1$ and $s \geq 2$, then the result follows from Theorem 2.11. For $r = 2$ and $s = 2$, $S_1 = \{e_{11}, e_{12}\}, S_2 = \{e_{11}, e_{21}\}, S_3 = \{e_{22}, e_{12}\}, S_4 = \{e_{22}, e_{21}\}$ are the χ' -sets of G such that $f_{\chi'}(S_i) = 2$ for $i = 1$ to 4 so that $f_{\chi'}(G) = 2$. So let $2 \leq r \leq s$. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the bipartite sets of G . Let $e_{1j} = x_1y_j(1 \leq j \leq s), e_{2j} = x_2y_j(1 \leq j \leq s), \dots, e_{ij} = x_iy_j(1 \leq i \leq r)(1 \leq j \leq s)$. Assign $c'(e_{1j}) = c'_j(1 \leq j \leq s), c'(e_{2j}) = c'_j, c'_1(2 \leq j \leq s), c'(e_{3j}) = c'_j, c'_1, c'_2(3 \leq j \leq s), \dots, c'(e_{ij}) = c'_s, c'_1, c'_2, \dots, c'_{s-1}(1 \leq i \leq r)(1 \leq j \leq s)$. Then $S_{ij} = \{e_{11}, e_{12}, e_{13}, \dots, e_{ks}\}$ is a χ' -set of G such that $\chi'(G) = s$. By Observation 2.3, $0 \leq f_{\chi'}(G) \leq s$. Since χ' -set of G is not unique $f_{\chi'}(G) \geq 1$. It is easily observed that no singleton subsets or two element subsets of

$S_{ij}(1 \leq i \leq r), (1 \leq j \leq s)$ is not a forcing subset of S_{ij} so that $f_{\chi'}(S_{ij}) = s$. Since this true for all χ' -set $S_{ij}(1 \leq i \leq r), (1 \leq j \leq s)$ so that $f_{\chi'}(G) = s$.

Theorem 2.14. For the path $G = P_n (n \geq 3)$, $f_{\chi'}(G) = \begin{cases} 0 & \text{if } n = 3 \\ 1 & \text{if } n = 4 \\ 2 & \text{if } n \geq 5 \end{cases}$

Proof. Let P_n be v_1, v_2, \dots, v_n and let $e_i = v_i v_{i+1} (1 \leq i \leq n-1)$. For $n = 3$, $S = E(G)$ is the unique χ' -set of G , the result follows from Theorem 2.8(a). For $n = 4$, $S_1 = \{e_1, e_2\}$ and $S_2 = \{e_2, e_3\}$ are the χ' -sets of G such that $f_{\chi'}(S_1) = f_{\chi'}(S_2) = 1$ so that $f_{\chi'}(G) = 1$. So let $n \geq 5$. Then $S_i = \{e_i, e_{i+1}\} (1 \leq i \leq n-1)$ and $S_{jk} = \{e_j, e_k\} (1 \leq j \leq k \leq n-1)$ and $|j-k|$ is odd are the only χ' -sets of G such that $f_{\chi'}(S_i) = 2$ for $(1 \leq i \leq n-1)$ and $f_{\chi'}(S_{jk}) = 2$ for $(1 \leq j \leq k \leq n-1)$ so that $f_{\chi'}(G) = 2$.

Theorem 2.15. For the cycle $G = C_n (n \geq 4)$, $f_{\chi'}(G) = 2$.

Proof. Let C_n be $v_1, v_2, \dots, v_n, v_1$ and let $e_i = v_i v_{i+1} (1 \leq i \leq n-1)$ and $e_n = v_n v_1$.

We consider the following two cases.

Case(1) n is even

$$c(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

Then $S_i = \{e_i, e_{i+1}\} (1 \leq i \leq n-1)$ and $S_{ijk} = \{e_j, e_k\} (1 \leq j \leq k \leq n-1)$ and $|j-k|$ is odd are the only χ' -sets of G such that $f_{\chi'}(S_i) = 2$ and $f_{\chi'}(S_{jk}) = 2$ for $(1 \leq i \leq n-1)$ and $(1 \leq j \leq k \leq n-1)$ so that $f_{\chi'}(G) = 2$.

Case(2) n is odd

$$c(e_i) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \\ 3 & \text{if } i = n \end{cases}$$

Since $E(v_n v_1)$ is the set of chromatic edges of G , $E(v_n v_1)$ is a subset of every χ' -set of G . It can be easily seen that any χ' -set of G is of the form $S = E(v_n v_1) \cup \{x, y\}$, where $x, y \in \{e_1, e_2, \dots, e_{n-1}\}$ so that $\chi'(G) = n + 2$. By Theorem 2.9, $f_{\chi'}(G) \leq n + 2 - n = 2$. Since χ' -set of G is not unique $f_{\chi'}(G) \geq 1$. It is easily observed that no singleton subsets of S is not a forcing subset of S so that $f_{\chi'}(S) = 2$. Since this is true for all χ' -set S of G , $f_{\chi'}(G) = 2$.

Theorem 2.16. For a positive integer $a \geq 2$, there exists a connected graph G such that $f_{\chi'}(G) = \chi'(G) = a$.

Proof. For $a = 2$, let $G = C_4$. Then by Theorem 2.15, $f_{\chi'}(G) = \chi'(G) = a$. So, let

$a \geq 3$. Let $G = K_{2,a}$. By Theorem 2.13, $f_{\chi'}(G) = \chi'(G) = a$.

3. Conclusion

In this article, we discuss about a new concept namely, forcing edge chromatic number of a graph. Also, the relation between edge chromatic number and forcing edge chromatic number is found. The above concept is examined by some standard graphs with examples.

4. References

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